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ON CLASSIFICATIONS FOR SOLUTIONS OF INTEGROQUASI-DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we have considered a quasi-differential expressions τ of order n with complex coefficients and its formal adjoint τ^+on$[0,b)$ respectively. We have shown in the case of one singular end-point and under suitable conditions on the integrand function $F\left(t,y,y^{[1]},\dots,y^{[n]},\ S(y)\right)$that all solutions of integroquasi-differential equation $[\tau-\lambda I]y(t)=wF$ are bounded and L_w^2 −bounded on$[0,b)$provided that all solutions of the equation $(\tau-\lambda I)y=0$ and its formal adjoint$\left(\tau^+-\bar{\lambda}I\right)v=0$ possess the same property, where $S(y)$ is the Sumudu transform of the function$y$.

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KEYWORDS: *Quasi-differential Expressions, Regular and Singular Endpoints, Minimal and Maximal Operators, Quasi-Differential Operators**,** Integro Quasi-differential Equations and their Solutions, Boundedness of Solutions**,** Sumudu Transform of the Function*

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1. INTRODUCTION

The problem that all solutions of a perturbed linear differential equation belong to $L_w^2(0,\infty)$ assuming the fact that all solutions of the unperturbed equation possess the same property considered by Wong,Zettland Everitt[1- 4]. In [5,6] S. E. Ibrahim extended their results for a general quasi-differential expression τ of arbitrary order n with complex coefficients, and considered the property of boundedness of solutions of a general integro quasi-differential equation

$$\tau[y]-\lambda wy=wf(t,y)(\lambda\epsilon\mathbb{C})\text{on}[0,b)\ , \tag{1.1}$$

where$f(t,y)$satisfies

$$|f(t,y)|\le k(t)+h(t)|y(t)|^{\sigma},\ \ t\epsilon[0,b)\text{for some } \sigma\in[0,1],$$

provided that all solutions of the equations:

$$\left(\tau-\lambda I\right)u=0\quad\text{and}\ \left(\tau^+-\bar{\lambda}I\right)v=0\ \ (\lambda\epsilon\mathbb{C}), \tag{1.2}$$

and their quasi-derivatives are in the space$L_w^2(0,b)$.

Our objective in this paper is to extend the results in [1 - 6] to more general class of integroquasi-differential equation in the form:

$$[\tau-\lambda I]y(t)=wF[t,y,y^{[1]},\dots,y^{[n]},S(y)] \quad \text{on}[0,b), \tag{1.3}$$

where $F[t,y,y^{[1]},\dots,y^{[n]},S(y)]$ satisfies

$$\left|F[t,y,y^{[1]},\dots,y^{[n]},S(y)]\right|\leq k(t)+h(t)\sum_{i=0}^{n}\left|S(y)y^{[i]}\right|^{\sigma}, t\in[0,b), 0<b\leq\infty \tag{1.4}$$

for some $\sigma\,\epsilon[0,1]$, $k(t)$, $h(t)$ are non-negative continuous functions on $[0,b)$ and $S(y)$ is the Sumudu transform of the function y. Also, we prove under suitable conditions on the function F that, all solutions of the equation (1.3) are bounded and L_w^2- bounded on the interval $[0,b)$ provided that all solutions of the equation $(\tau-\lambda I)y=0$ and its formal adjoint $(\tau^+-\bar{\lambda}I)v=0$ possess the same property, where τ^+ is the formal adjoint of τ.

2. SumuduTransform and Some Technical Lemmas

The Sumudu transform method (STM) was in part re-initiated in 1993 by Watugala [7-9]who used it to solve engineering control problems. The first application of the inverse formula was done by Weerakoon [10]. Sumudu transformbased solutions to convolution type integral equations and discrete dynamic systems were later obtained by Asiru [11-13]. Subsequently, it expanded to two variables in [14].

Definition 2.1 (cf. [7-14]): The Sumudu transform is defined for possibly bilateral functions in the set,

$$A=\{f(t)\mid\exists K,t_1,t_2>0,\ |f(t)|<Ke^{\frac{|t|}{t_i}},\ \text{if}\, t\in(-1)^i\times[0,\infty)\}, \tag{2.1}$$

by the following integration,

$$F(u)=S[f(t)]=\int_0^\infty f(ut)e^{-t}\,dt,\ u\in(-t_1,t_2). \tag{2.2}$$

Remark: Note that by considering for instance, $f(t)=e^t$,then $f(ut)=e^{ut}$ and hence,the Sumudu transform (Right side, t is non-negative) of the function f is then,

$$S[f(t)]=\int_0^\infty e^{ut}e^{-t}\,dt,$$

$$=\lim_{b\to\infty}\int_0^b e^{ut}e^{-t}\,dt$$

$$=\lim_{b\to\infty}\int_0^b e^{(u-1)t}\,dt$$

$$=\lim_{b\to\infty}\frac{e^{(u-1)t}}{u-1}\Big|_0^b=\frac{1}{1-u},\quad 0\leq u<1. \tag{2.3}$$

If $u\geq 1$, then the previous integral will be divergent.

The Sumudu transform of the first derivative of the function $f(t)$, $f'(t)=df(t)/dt$ is given by:

$$\mathrm{S}\left[\frac{df(t)}{dt}\right]=\frac{1}{u}[F(u)-f(0)]. \tag{2.4}$$

The Sumudu transform of the second derivative of $f(t)$, $f''(t)=d^2f(t)/dt^2$ is given by:

$$S\left[\frac{d^2f(t)}{dt^2}\right] = \frac{1}{u^2}\left[F(u) - f(0) - u\frac{df(t)}{dt}\ |_{t=0}\right]. \tag{2.5}$$

Theorem 2.2 (cf. [14]): If $F(u)$ is Sumudu transform of $f(t)$, then the Sumudu transform of any integer n-order derivative of $f(t)$, $f^{(n)}(t) = d^n f(t)/dt^n$ is given by:

$$S\left[\frac{d^nf(t)}{dt^n}\right] = u^{-n}\left[F(u) - f(0) - \sum_{k=0}^{n-1} u^k\frac{d^kf(t)}{dt^k}\ |_{t=0}\right]. \tag{2.6}$$

In the sequel we shall require the following nonlinear integral inequality which generalizes those integral inequalities used in [1- 6] and [15 - 22].

Lemma 2.3: Gronwall' s Inequality (cf. [1-6] and [15-17]): Let $u(t)$ and $v(t)$ be two non-negative continuous functions on the interval $I = [0, b)$, $c \geq 0$ be a constant. The classical Gronwall's inequality states that: if

$$u(t) \leq c + \int_0^t v(s)u(s)dx\,, \qquad 0 \leq t \leq 1.$$

Then

$$u(t) \leq c\exp\left(\int_0^t v(s)ds\right)ds. \qquad 0 \leq t \leq 1. \tag{2.7}$$

Lemma 2.4: (cf. [1-6] and [21, 22]): Let $u(t)$ and $v(t)$ be two non-negative continuous functions and locally integrable on the interval$I = [0, b)$, $\sigma\epsilon\,[0,1]$. Then the inequality

$$u(t) \leq c_0 + \int_0^t v(s)u^{\sigma}(s)dx, \qquad c_0 > 0.$$

For $0 \leq \sigma < 1$, implies that

$$u(t) \leq \left((c_0)^{(1-\sigma)} + (1-\sigma)\int_0^t v(s)ds\right)^{\frac{1}{(1-\sigma)}}ds. \tag{2.8}$$

In particular, if $v(s) \in L^1(0, b)$, then (2.8) implies that $u(t)$ is bounded.

Lemma 2.5: (cf. [1-6], [21, 22]): Let $u(t)$, $z(t)$, $g(t)$ and $h(t)$ be non-negative continuous functions defined on the interval$I = [0, b)$ and suppose that the inequality

$$u(t) \leq z(t) + g(t)\left(\int_0^t u^2(s)h(s)dx\right)^{\frac{1}{2}}\text{for } t \geq 0\,.$$

Then

$$u(t) \leq z(t) + g(t)\left(\int_0^t 2z^2(s)h(s)exp\left[\int_0^s 2g^2(x)h(x)dx\right]ds\right)^{\frac{1}{2}}, \text{ for } t \geq 0. \tag{2.9}$$

3. Quasi-Differential Expressions

The quasi-differential expressions are defined in terms of a Shin-Zettl matrix A on an interval I. The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $n \times n$-matrices $A = \{a_{rs}\}$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$a_{rs}\epsilon L^2_{loc}(I), \qquad (1 \leq r, s \leq n, n \geq 2)$$

$$a_{r,r+1} \neq 0, \qquad a.\,e., \text{ on } I, \quad (1 \leq r \leq n-1) \tag{3.1}$$

$a_{rs} = 0, a.\ e.,$ on $I,\quad (2 \le r+1 < s \le n).$

For$A \epsilon Z_n(I)$,the quasi-derivatives associated with A are defined by:

$$y^{[0]} := y,$$

$$y^{[r]} := (f_{r,r+1})^{-1}\left\{\left(y^{[r-1]}\right)' - \sum_{s=1}^{r} a_{rs}\, y^{[s-1]}\right\}, (1 \le r \le n-1), \tag{3.2}$$

$$y^{[n]} := \left\{\left(y^{[n-1]}\right)' - \sum_{s=1}^{n} a_{ns}\, y^{[s-1]}\right\},$$

where the prime $'$ denotes differentiation .

The quasi-differential expression τ associated with A is given by

$$\tau[.] := i^n y^{[n]}, \quad (n \ge 2), \tag{3.3}$$

this being defined on the set :

$$V(\tau) := \{y: y^{[r-1]} \epsilon AC_{loc}(I), \quad r = 1,2, \dots, n\},$$

where $AC_{loc}(I)$,denotes the set of functions which are absolutely continuous on every compactsubinterval of I.

The formal adjoint τ^+ofτ is defined by the matrix A^+ given by:

$$\tau^+[.] := i^n y_+^{[n]}, \quad \text{for all} \quad y \epsilon V(\tau^+), \tag{3.4}$$

$$V(\tau^+) := \left\{y: y_+^{[r-1]} \epsilon AC_{loc}(I), r = 1,2, \dots, n\right\},$$

where $y_+^{[r-1]}$, the quasi-derivatives associated with the matrixA^+ in$Z_n(I)$,

$$A^+ = (a_{rs})^+ = (-1)^{r+s+1}\, \overline{a_{n-s+1,n-r+1}}\ , \tag{3.5}$$

for each r and s .

Note that:$(A^+)^+ = A$ andso$(\tau^+)^+ = \tau$. We refer to [4 -6], [15] and [18-20] for a full account of the above and subsequent results on quasi-differential expressions.

For u $\epsilon V(\tau)$, $v \epsilon V(\tau^+)$and $\alpha, \beta \epsilon I$, we have the Green's formula

$$\int_a^b \left\{\bar{v}\tau[u] - u\overline{\tau^+[v]}\right\} dx = [u,v](b) - [u,v](a), \tag{3.6}$$

where

$$[u,v](x) = i^n \left(\sum_{r=0}^{n-1} (-1)^{r+s+1} u^{[r]}(x)\overline{v_+^{[n-r-1]}}(x)\right)$$

$$= (-i)^n \left(u, u^{[1]}, \dots, u^{[n-1]}\right) \times J_{n\times n} \times \begin{pmatrix} \bar{v} \\ \vdots \\ \bar{v}_+^{[n-1]} \end{pmatrix}(x); \tag{3.7}$$

see [1], [5, 6], [15] and [20].

Let the interval I have end-points $a,b(-\infty \leq a < b \leq \infty)$,and let $w : I \to \mathbb{R}$ be a non-negative weight function with $w\epsilon L^1_{loc}(I)$and $w > 0$ (for almost all $x\epsilon I$). Then $H = L^2_w(I)$denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that$\int_I w|f|^2 < \infty$; the inner-product is defined by:

$$(f,g) := \int_I f(x)\overline{g(x)}dx \ (f,g\epsilon L^2_w(I). \quad (3.8)$$

The equation

$$\tau[u] - \lambda wu = 0(\lambda\epsilon\mathbb{C})onI, \quad (3.9)$$

is said to be **regular** at the left end-point$a\epsilon\mathbb{R}$, if for all $X\epsilon\ (a,b)$,$a\epsilon\mathbb{R}$, $w,\ f_{rs}\epsilon L^1(a,X),\ (r,s = 1,2,\dots,n)$.

Otherwise (3.9) is said to be **singular** at a **.** If (3.9) is regular at both end-points, then it is said to be regular; in this case, we have

$$a,b\epsilon\mathbb{R}, \quad w,f_{rs}\epsilon L^1(a,b), \ (r,s = 1,2,\dots,n).$$

We shall be concerned with the case when a is a regular end-point of (3.9), the end-point $b$being allowed to be either regular or singular. Note that, in view of (3.5), an end-point of I is regular for (3.9), if and only if it is regular for the equation

$$\tau^+[v] - \overline{\lambda}wv = 0\ (\lambda\epsilon\mathbb{C}) \ \text{on} \ I. \quad (3.10)$$

Note that, at a regular end-point$a$, say,$u^{[r-1]}(a)\left(v_+^{[r-1}(a)\right)$,$r = 1,2,\dots,n$ is defined for all $u\epsilon V(\tau)\left(v\epsilon V(\tau^+)\right)$.

Denote by $S(\tau)$ and $S(\tau^+)$ the sets of all solutions of the equations

$$\left(\tau - \lambda_0 I\right)u = 0 \quad \text{and}(\tau^+ - \overline{\lambda_0}I)v = 0 \quad (3.11)$$

respectively, and let $S^r(\tau) = \left\{y^{[r]}:\ (\tau - \lambda_0 I)y = 0,\ r = 1,2,\dots,n-1\right\}$ denotethe set of all quasi-derivatives of solutions of the equation $(\tau - \lambda_0 I)u = 0$.Let $\varphi_k(t,\lambda),\ k = 1,2,\dots,n$be the solutions of the homogeneous equation

$$\left(\tau - \lambda I\right)u = 0\ (\lambda\epsilon\mathbb{C}), \quad (3.12)$$

satisfying

$$\varphi_j^{[k-1]}(t_0,\lambda) = \delta_{k,r+1}\text{for all}t_0\epsilon[a,b), \qquad (j,k = 1,2,\dots,n,\ r = 0,1,\dots,n-1),$$

for fixedt_0, $a < t_0 < b$. Then $\varphi_j^{[r]}(t,\lambda)$ is continuous in (t,λ) for $a < t < b$, $|\lambda| < \infty$,and for fixed t it is entire in λ .Let $\varphi_k^+(t,\lambda),\ k = 1,2,\dots,n$ denote the solutions of the adjoint homogeneous equation

$$\left(\tau^+ - \overline{\lambda}I\right)v = 0\ (\lambda\epsilon\mathbb{C}), \quad (3.13)$$

satisfying

$$(\varphi_k^+)^{[r]}(t_0,\lambda) = (-1)^{k+r}\delta_{k,n-r}\text{for all } t_0\epsilon[0,b),\ (k = 1,2,\dots,n,\ r = 0,1,\dots,n-1).$$

Suppose$0 < c < b$.By [5, 6] and [15], a solution of the equation

$$(\tau - \lambda I)u = wf(\lambda\epsilon\mathbb{C}), \qquad f\epsilon L^1_w(0,b), \quad (3.14)$$

satisfying$u^{[r]}(c) = 0,\ \ r = 0,1,\dots,n-1$is giving by

$$\varphi(t,\lambda) = ((\lambda-\lambda_0)/i^n)\sum_{j,k=1}^{n}\xi^{jk}\,\varphi_j(t,\lambda)\int_a^t \overline{\varphi_k^+(s,\lambda)}\,f(s)w(s)ds,$$

where$\varphi_k^+(t,\lambda)$stands for the complex conjugate of $\varphi_k(t,\lambda)$ and for each j,k,ξ^{jk} is constant which is independent of t, λ (but does depend in general on t).

The next lemma is a form of the variation of parameters formula for a general quasi-differential equation is giving by the following Lemma.

Lemma 3.1: (cf. [5, 6, 15]): Suppose $f\epsilon L_w^1(0,b)$ locally integrable function and $\varphi(t,\lambda)$is the solution of the equation (3.14) satisfying:

$$\varphi^{[r]}(t_0,\lambda) = \alpha_{r+1}\ \ \text{for}\ r = 0,1,\dots,n-1, t_0\epsilon[0,b).$$

Then

$$\varphi(t,\lambda) = \sum_{j=1}^{n}\alpha_j(\lambda)\varphi_j(t,\lambda_0) + ((\lambda-\lambda_0)/i^n)\sum_{j,k=1}^{n}\xi^{jk}\varphi_j(t,\lambda_0)\int_a^t\overline{\varphi_k^+(s,\lambda_0)}\,f(s)w(s)ds. \tag{3.15}$$

for some constants $\alpha_1(\lambda), \alpha_2(\lambda), \dots, \alpha_n(\lambda)\epsilon\mathbb{C}$, where $\phi_j(t,\lambda_0)$ and $\varphi_k^+(s,\lambda_0)$, $j,k = 1,2,\dots,n$ are solutions of the equations in (3.11) respectively,$\xi^{jk}$is a constant which is independent of t.

Proof: The proof is similar to that in [5,6], [15] and [18-20] for more details.

Remark: Lemma 3.1contains the following lemma as a special case.

Lemma 3.2: Suppose $f\epsilon L_w^1(0,b)$ locally integrable function and $\phi(t,\lambda)$ is the solution of the equation (3.14) satisfying:

$$\varphi^{[r]}(t_0,\lambda) = \alpha_{r+1}\text{for}r = 0,1,\dots,\ n-1, t_0\epsilon[a,b).$$

Then

$$\varphi^{[r]}(t,\lambda) = \sum_{j=1}^{n}\alpha_j(\lambda)\varphi_j^{[r]}(t,\lambda_0) + \frac{1}{i^n}(\lambda-\lambda_0)\sum_{j,k=1}^{n}\xi^{jk}\varphi_j^{[r]}(t,\lambda_0)\int_a^t\overline{\varphi_k^+(t,\lambda_0)}\,f(s)w(s)ds, \tag{3.16}$$

for $r = 0,1,\dots,n-1$.

Proof: The proof follows from Lemma 3.1 and on applying the r^{th} quasi-derivatives on both sides of the equation (3.15). We refer to [4 -6] and [15-18] for more details.

Lemma 3.3: Suppose that for some $\lambda_0\epsilon\mathbb{C}$all solutions of the equations in (3.11) are in $L_w^2(0,b)$.Then all solutions of the equations (3.12) and (3.13) are in $L_w^2(0,b)$for every complex number $\lambda\epsilon\mathbb{C}$.

Proof: The proof is similar to that in [4 - 6], [15-18] and [20].

Lemma 3.4: If all solutions of the equation $(\tau-\lambda_0 w)u = 0$ are bounded on [0,b) and$\varphi_k^+(t,\lambda_0) \in L_w^1(0,b)$ for some $\lambda_0\epsilon\mathbb{C}$, $k = 1,\dots,n$. Then all solutions of the equation$(\tau-\lambda w)u = 0$ are also bounded on [0,b) for every complex number $\lambda\epsilon\mathbb{C}$.

Lemma 3.5: Suppose that for some complex number $\lambda_0\epsilon\mathbb{C}$ all solutions of the equations in (3.11) are in $L_w^2(0,b)$. Suppose $f\epsilon L_w^2(0,b)$, then all solutions of the equation (3.14) are in $L_w^2(0,b)$for all $\lambda\epsilon\mathbb{C}$.

Proof: Let $\{\varphi_1(t,\lambda), \varphi_2(t,\lambda), \dots, \varphi_n(t,\lambda)\}$, $\{\varphi_1^+(s,\lambda), \varphi_2^+(s,\lambda), \dots, \varphi_n^+(s,\lambda)\}$ be two sets of linearly independent solutions of the equations (3.11). Then for any solutions $\phi(t, \lambda)$ of the equation $(\tau-\lambda I)\varphi = wf(\lambda\epsilon\mathbb{C})$ which may be written as follows $(\tau - \lambda_0 w)\varphi = (\lambda - \lambda_0)w\varphi + wf$ and it follows from (3.15) that

$$\varphi(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j(t,\lambda_0) + \frac{1}{i^n}\sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t,\lambda_0)\int_a^t \overline{\varphi_k^+(t,\lambda_0)}\,[(\lambda-\lambda_0)\varphi(s,\lambda) + f(s)]w(s)ds, \quad \textbf{(3.17)}$$

for some constants $\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_n(\lambda)\epsilon\mathbb{C}$. Hence

$$|\varphi(t,\lambda)| = \sum_{j=1}^{n}\left(|\alpha_j(\lambda)||\varphi_j(t,\lambda_0)|\right) + \sum_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)|$$

$$\times\int_a^t \overline{\varphi_k^+(t,\lambda_0)}\,[|\lambda-\lambda_0||\varphi(s,\lambda)| + |f(s)|]w(s)ds. \quad \textbf{(3.18)}$$

Since $f\epsilon L_w^2(0,b)$ and $\varphi_k^+(.,\lambda_0)\epsilon L_w^2(a,b)$ for some $\lambda_0\epsilon\mathbb{C}$, then $\varphi_k^+(.,\lambda_0)f\epsilon L_w^1(a,b)$, for some $\lambda_0\epsilon\mathbb{C}$ and $k = 1,\ldots,n.$ Setting

$$C_j(\lambda) = \sum_{j,k=1}^{n}|\xi^{jk}|\int_a^b\left|\overline{\varphi_k^+(t,\lambda_0)}\right||f(s)|w(s)ds,\ \ j = 1,2,\ldots,n, \quad (3.19)$$

then

$$|\varphi(t,\lambda)| \le \sum_{j=1}^{n}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)|\varphi_j(t,\lambda_0)|$$

$$+\,|\lambda-\lambda_0|\sum_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)|\int_a^b\left|\overline{\varphi_k^+(t,\lambda_0)}\right||\varphi(s,\lambda)||f(s)|w(s)ds. \quad (3.20)$$

On application of the Cauchy-Schwartz inequality to the integral in (3.18), we get

$$|\varphi(t,\lambda)| \le \sum_{j=1}^{n}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)|\varphi_j(t,\lambda_0)|$$

$$+\,|\lambda-\lambda_0|\sum_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)|\left(\int_a^b\left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)ds\right)^{\frac{1}{2}}\left(\int_0^b|\varphi(s,\lambda)|^2w(s)ds\right)^{\frac{1}{2}}. \quad (3.21)$$

By using the inequality $(u+v)^2 \le 2(u^2+v^2)$, it follows that

$$|\varphi(t,\lambda)|^2 \le 4\sum_{j=1}^{n}(C_j^2 + |\alpha_j(\lambda)|^2)|\varphi_j(t,\lambda_0)|^2 + \ 4|\lambda-\lambda_0|^2 \times$$

$$\sum_{j,k=1}^{n}|\xi^{jk}|^2|\varphi_j(t,\lambda_0)|^2\left(\int_a^b\left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)ds\right)\left(\int_0^b|\varphi(s,\lambda)|^2w(s)ds\right). \quad (3.22)$$

By hypothesis there exist positive constant K_0 and K_1 such that

$$\|\varphi_j(t,\lambda_0)\|_{L_w^2(0,b)} \le K_0 \text{ and } \left\|\overline{\varphi_k^+(s,\lambda_0)}\right\|_{L_w^2(0,b)} \le K_1;\ j,k = 1,2,\ldots,n\,. \quad (3.23)$$

Hence

$$|\varphi(t,\lambda)|^2 \le 4\sum_{j=1}^{n}(C_j^2 + |\alpha_j(\lambda)|^2)|\varphi_j(t,\lambda_0)|^2$$

$$+\ 4K_1^2|\lambda-\lambda_0|^2\sum_{j,k=1}^{n}|\xi^{jk}|^2|\varphi_j(t,\lambda_0)|^2\left(\int_0^b|\varphi(s,\lambda)|^2w(s)ds\right). \quad (3.24)$$

By integrating the inequality in (3.24) between 0 and t, we obtain

$$\int_0^t|\varphi(s,\lambda)|^2\,w(s)ds \le K_2 + \left(4|\lambda-\lambda_0|^2\sum_{j,k=1}^{n}|\xi^{jk}|^2\right)\int_0^t|\varphi_j(t,\lambda_0)|^2\left(\int_0^s|\varphi(x,\lambda)|^2w(x)dx\right)w(s)ds, \quad (3.25)$$

where

$$K_2 = 4K_0^2 \sum_{j=1}^{n^2N} \left(C_j^2 + \left|\alpha_j(\lambda)\right|^2\right). \tag{3.26}$$

Now, on using Gronwall' s inequality, it follows that

$$\int_0^t |\varphi(s,\lambda)|^2 w(s)ds \le K_2 \, exp\left(4K_1^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n}\left|\xi^{jk}\right|^2 \int_0^t \left|\varphi_j(t,\lambda_0)\right|^2 w(s)ds\right). \tag{3.27}$$

Since, $\varphi_j(t,\lambda_0)\epsilon L_w^2(a,b)$ for some $\lambda_0 \epsilon \mathbb{C}$ and for $j = 1,\dots,n$, then $\phi(t,\lambda)\epsilon L_w^2(0,b)$.

Remark: Lemma 3.5 also holds if the function f is bounded on $[0,b)$.

Lemma 3.6: Let $f\epsilon L_w^2(0,b)$. Suppose for some $\lambda_0 \epsilon \mathbb{C}$ that:

(i) All solutions of $(\tau^+ - \bar{\lambda} I)\,\varphi^+ = 0$ are in $L_w^2(a,b)$.

(ii) $\varphi_j^{[r]}(t,\lambda_0)$, $j = 1,\dots,n$ are bounded on $[0,\mathrm{b})$ for some $r = 0,1,\dots,n-1$.

Then $\varphi^{[r]}(t,\lambda)\epsilon L_w^2(0,b)$ for any solution $\phi(\mathrm{t},\lambda)$ of the equation $(\tau - \lambda I)\varphi = wf$, for all $\lambda \epsilon \mathbb{C}$.

Proof: The proof is the same up to (3.20). By using Lemma 3.2,(3.20) becomes,

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^{n}\left(\left|\alpha_j(\lambda)\right| + C_j(\lambda)\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right| + |\lambda-\lambda_0|$$

$$\times \sum_{j,k=1}^{n}\sum_{r=0}^{n-1}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right| \int_a^b \left|\overline{\varphi_k^+(t,\lambda_0)}\right| \left|\varphi^{[r]}(t,\lambda))\right| w(s)ds. \tag{3.28}$$

On applying the Cauchy-Schwartz inequality to the integral in (3.28),we get

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^{n}\left(C_j + \left|\alpha_j(\lambda)\right|\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right| + |\lambda-\lambda_0| \sum_{j,k=1}^{n}\sum_{r=0}^{n-1}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

$$\times \left(\int_0^t \left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)ds\right)^{\frac{1}{2}} \left(\int_0^t \left|\varphi^{[r]}(t,\lambda))\right|^2 w(s)ds\right)^{\frac{1}{2}}, \tag{3.29}$$

By using the inequality $(u+v)^2 \le 2(u^2+v^2)$,it follows that

$$\left|\varphi^{[r]}(t,\lambda)\right|^2 \le 4\sum_{j=1}^{n}\left(C_j^2 + \left|\alpha_j(\lambda)\right|^2\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right|^2 + 4|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n}\sum_{r=0}^{n-1}\left|\xi^{jk}\right|^2\left|\varphi_j^{[r]}(t,\lambda_0)\right|^2$$

$$\times \left(\int_0^t \left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)ds\right)\left(\int_0^t \left|\varphi^{[r]}(s,\lambda)\right|^2 w(s)ds\right), \tag{3.30}$$

Since $\varphi_k^+(t,\lambda_0)\epsilon L_w^2(0,b)$ for some $\lambda_0 \in \mathbb{C}$ and $\varphi_j^{[r]}(t,\lambda_0), j = 1,\dots,n$ are bounded on $[0,\mathrm{b})$ for some $r = 0,1,\dots,n-1$ by hypothesis, then there exist a positive constants K_0 and K_1 such that

$$\left|\varphi_j^{[r]}(t,\lambda_0)\right| \le K_0 \text{and} \left\|\overline{\varphi_k^+(s,\lambda_0)}\right\|_{L_w^2(0,b)} \le K_1. \tag{3.31}$$

Hence,

$$\left|\varphi^{[r]}(t,\lambda)\right|^2 \le 4K_0^2 \sum_{j=1}^{n}\left(C_j^2 + \left|\alpha_j(\lambda)\right|^2\right) + 4K_0^2K_1|\lambda-\lambda_0|^2$$

$$\times \sum_{j,k=1}^{n}\sum_{r=0}^{n-1}\left|\xi^{jk}\right|^2\left(\int_0^t \left|\varphi^{[r]}(s,\lambda)\right|^2 w(s)ds\right). \tag{3.32}$$

By integrating the inequality in (3.32) between 0 and t, and by using Lemma2.3 (Gronwall' s inequality),we have the result.

4. Boundedness and L_w^2 – Solutions

In this section, we shall consider the question of determining conditions under which all solutions of the equation (1.3) are bounded andL_w^2 – bounded.

Suppose there exist non-negative continuous functions $k(t)$ and $h(t)$on$[0,b)$, $0 < b \le \infty$; such that

$$\left|F[t,y,y^{[1]},\dots,y^{[n]},S(y)]\right| \le k(t) + h(t)\sum_{i=0}^{n}\left|S(y)y^{[i]}\right|^{\sigma}, \quad t \in [0,b), \tag{4.1}$$

forsome $\sigma \epsilon [0,1]$, $-\infty < y^{[i]} < \infty$, for each $i = 0,1,\dots,n-1$;see [1 - 6] and [15].

Theorem 4.1: Suppose that the function F satisfies (4.1) with $\sigma = 1$, $S^r(\tau) \cup S(\tau^+) \subset L^\infty(0,b)$ for some $r = 0,1,\dots,n-1$, for some $\lambda_0 \in \mathbb{C}$and that

$(i) k(t) \in L_w^1(0,b)$for all $t \in [0.b)$,

$(ii) h_i(t) \in L_w^1(0,b)$for all $t \in [0.b)$, $i = 0,1,\dots,n-1$.

Then$\varphi^{[r]}(t,\lambda)$, $r = 0,1,\dots,n-1$ are bounded on $[0,b)$ for any solutions $\varphi(t,\lambda)$ of the equation (1.3), for all$\lambda \in \mathbb{C}$.

Proof: Note that (4.1) and Lemma 3.6 implies that all solutions are defined on $[0,\mathrm{b})$, see [1-6], [15] and [20, Chapter 3]. Let $\{\varphi_1(t,\lambda_0), \varphi_2(t,\lambda_0),\dots,\varphi_n(t,\lambda_0)\}$, $\{\varphi_1^+(s,\lambda_0),\varphi_2^+(s,\lambda_0),\dots,\varphi_n^+(s,\lambda_0)\}$ be two sets of linearly independent solutions of the equations (3.11) respectively, and let $\varphi(t,\lambda)$ be any solution of the equation (1.3) on $[0,b)$, then by Lemma 3.2, we have

$$\varphi^{[r]}(t,\lambda) = \sum_{j=1}^{n}\alpha_j(\lambda)\varphi_j^{[r]}(t,\lambda_0) + \frac{1}{i^n}(\lambda-\lambda_0)\sum_{j,k=1}^{n}\xi^{jk}\varphi_j^{[r]}(t,\lambda_0)$$

$$\times \int_a^t \overline{\varphi_k^+(s,\lambda_0)}\, F[t,y,y^{[1]},\dots,y^{[n]},S(\varphi(t))]w(s)ds, \text{for } r = 0,1,\dots,n-1. \tag{4.2}$$

Hence

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^{n}\left|\alpha_j(\lambda)\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right| + |\lambda-\lambda_0|\sum_{j,k=1}^{n}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right|\int_a^t\left|\overline{\varphi_k^+(s,\lambda_0)}\right|$$

$$\times \int_0^t\left|\overline{\varphi_k^+(s,\lambda_0)}\right|\,[k(s) + \sum_{i=0}^{n-1}h_i(s)\left|S(\varphi(s))\varphi^{[i]}\right|]\,w(s)ds, r = 0,1,\dots,n-1. \tag{4.3}$$

Since $k(s) \in L_w^1(0,b)$ and $\varphi_k^+(s,\lambda_0)$, $k = 1,2,\dots,n$ are bounded on $[0,b)$ for some $\lambda_0 \in \mathbb{C}$, we have $\varphi_k^+(s,\lambda_0)k(s) \in L_w^1(0,b)$, $k = 1,2,\dots,n$ for some $\lambda_0 \in \mathbb{C}$.Setting

$$C_j = |\lambda-\lambda_0|\sum_{j,k=1}^{n}\left|\xi^{jk}\right|\int_a^t\overline{\varphi_k^+(t,\lambda_0)}\,k(s)w(s)ds, \quad j = 1,2,\dots,n. \tag{4.4}$$

Then by (2.3)

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^{n}\left(C_j + |\alpha_j(\lambda)|\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right| + \frac{1}{|1-u|}|\lambda-\lambda_0|\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

$$\times \int_0^t\left|\overline{\varphi_k^+(t,\lambda_0)}\right|\,h_i(s)\left|\varphi^{[i]}(s,\lambda)\right|w(s)ds, \quad r = 0,1,\dots,n-1. \tag{4.5}$$

By hypothesis, there exist a positive constants K_0 andK_1 such that

$\left|\varphi_j^{[r]}(t,\lambda_0)\right| \le K_0and|\varphi_k^{+}(t,\lambda_0)| \le K_1$for all $t \in [0,b);\quad j,k = 1,\dots,n,\ r = 0,1,\dots,n-1.$

Hence, by summing both sides of(4.5) from $r = 0$to$n-1$,we get

$$\sum_{r=0}^{n-1}\left|\varphi^{[r]}(t,\lambda)\right| \le (n-1)K_0\sum_{j=1}^{n}\left(C_j + \left|\alpha_j(\lambda)\right|\right) + (n-1)K_0K_1|\lambda-\lambda_0|\sum_{j,k=1}^{n}\left|\xi^{jk}\right|$$

$$\times \int_0^t \left(max_{0\le i\le(n-1)}h_i(s)\right)\left(\sum_{i=0}^{n-1}\left|\varphi^{[i]}(s,\lambda)\right|\right)w(s)dx. \tag{4.6}$$

On applying Gronwall' s inequality to (4.6) and by using (ii), we deduce that$\sum_{r=1}^{n-1}\left|\varphi^{[r]}(t,\lambda)\right|$ is finite and hence the result.

Remark: From [3, Section 3] and [4], φ and$\varphi^{[j]} \in L_w^1(0,b)$ implies that$\varphi^{[r]}(t,\lambda) \in L_w^1(0,b)$ for any solution $\varphi(t,\lambda)$of the equation (1.3) for all $\lambda \in \mathbb{C},\ r = 1,\dots,j-1,\ 1 \le j \le n-1.$

Theorem 4.2: Suppose that the function F satisfies (4.1) with $\sigma = 1$, $S^r(\tau) \cup S(\tau^+) \subset L_w^2(0,b)$, for some $\lambda_0 \in \mathbb{C}$ and some $r = 0,1,\dots,n-1$, and that

$(i)k(t) \in L_w^2(0,b)$for all$t \in [0.b)$,

$(ii)h_i(t) \in L_w^2(0,b)$for all$t \in [0.b),\quad i = 0,1,2,\dots,n-1.$

Then$\varphi^{[r]}(t,\lambda) \in L_w^2(0,b),\ r = 0,1,\dots,n-1$ for any solutions $\varphi(t,\lambda)$ of the equation (1.3), for all$\lambda \in \mathbb{C}$.

Proof: Applying the Cauchy-Schwartz inequality to the integral in (4.5) we get,

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^{n}\left(C_j + \left|\alpha_j(\lambda)\right|\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right| + \frac{1}{|1-u|}|\lambda-\lambda_0|\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

$$\times\left(\int_0^t\left|\overline{\varphi_k^{+}(t,\lambda_0)}\right|^2|h_i(s)|w(s)ds\right)^{\frac{1}{2}}\left(\int_0^t|h_i(s)|\left|\varphi^{[i]}(s,\lambda)\right|^2w(s)ds\right)^{\frac{1}{2}},\ r = 0,1,\dots,n-1. \tag{4.7}$$

Since$\varphi_k^{+}(t,\lambda_0)\epsilon L_w^2(0,b)$,for some $\lambda_0 \in \mathbb{C}$ and $h_i(t) \in L^{\infty}(0,b)$ by hypothesis, then$\varphi_k^{+}(t,\lambda_0)|h_i(t)|^{\frac{1}{2}}\epsilon L_w^2(0,b), k = 1,2,\dots,n,\ i = 0,1,\dots,n-1.$ Let,

$$D_{ki} = \left(\int_0^t\left|\overline{\varphi_k^{+}(t,\lambda_0)}\right|^2|h_i(s)|w(s)ds\right)^{\frac{1}{2}}, z(t) = \sum_{j=1}^{n}\left(C_j + \left|\alpha_j(\lambda)\right|\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

and

$$G(t) = \frac{1}{|1-u|}|\lambda-\lambda_0|\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right|.$$

From Lemma 2.5 we have

$$\left|\varphi^{[r]}(t,\lambda)\right| \le Z(t) + G(t)\left(\int_0^t 2Z^2(s)\,|h_i(s)|exp\left[\int_0^s 2G^2(x)|h_i(x)|w(x)dx\right]w(s)ds\right)^{\frac{1}{2}}.$$

Since$\int_0^t Z^2(s)\,|h_i(s)|w(s)dsand\int_0^s G^2(x)|h_i(x)|w(x)dx$are both finite, we conclude that $\varphi^{[r]}(t,\lambda)$is bounded by a linear combination of $L_w^2(0,b)$ functions $Z(t)$and $G(t)$.Therefore, by using Lemma 2.5,$\varphi^{[r]}(t,\lambda) \in L_w^2(0,b),\ r =$

$0,1,\ldots n-1$for all $\lambda \in \mathbb{C}$.

Remark: If we use the Cauchy-Schwartz inequality for the integral in (4.5) as:

$$\int_0^t \left|\overline{\varphi_k^+(t,\lambda_0)}\right| |h_i(s)| \left|\varphi^{[i]}(s,\lambda)\right| w(s)ds \leq \left(\int_0^t \left|\overline{\varphi_k^+(s,\lambda_0)}\right|^2 |h_i(s)|^2 w(s)ds\right)^{\frac{1}{2}} \left(\int_0^t \left|\varphi^{[i]}(s,\lambda)\right|^2 w(s)ds\right)^{\frac{1}{2}},$$

$i = 0,1,\ldots,n-1$, we also get the result. We refer to [1 - 3] for more details.

Corollary 4.3: Suppose that$|F(t,y(t),S(y))| = k(t) + h(t)|S(y)|$, $S^r(\tau) \cup S(\tau^+) \subset L_w^2(0,b)$ for some $\lambda_0 \in \mathbb{C}$,and that$h(t)\epsilon L_w^p(0,b)$ for some $p \geq 2$, $t \in [0,b)$. Then $\varphi^{[r]}(t,\lambda) \in L_w^1(0,b)$ for any solutions $\varphi(t,\lambda)$ of the equation (1.3), for all$\lambda \in \mathbb{C}$and all$r = 0,1,\ldots,n-1$.

Corollary 4.4: Suppose that for some $\lambda_0\epsilon\mathbb{C}$, if all solutions of the equations $[\tau]u = \lambda_0 wu$ and$[\tau^+]v = \overline{\lambda_0}wv$ are in the space $L_w^2(0,b)$for some$\lambda_0 \in \mathbb{C}$ and $k(t) \in L_w^2(0,b)$. Then all solutions of the equations $[\tau - \lambda w]\varphi = wk$ are in the space$L_w^2(0,b)$for every complex number $\lambda\,\epsilon\mathbb{C}$.

Next, for considering (4.1) with$0 \leq \sigma < 1$,we have the following.

Theorem 4.5: Suppose that F satisfies (4.1) with$0 \leq \sigma < 1$,$S^r(\tau) \cup S(\tau^+) \subset L_w^2(0,b)$ for some $\lambda_0 \in \mathbb{C}$ and some$r = 0,1,\ldots,n-1$, and that

$(i) k(t) \in L_w^2(0,b)$for all$t \in [0,b)$,

(ii) $h_i(t) \in L_w^{2/(1-\sigma)}(0,b)$forall $t \in [0,b)$, $i = 0,1,\ldots,n-1$.

Then$\varphi^{[r]}(t,\lambda) \in L_w^2(0,b)$, $r = 0,1,\ldots,n-1$ for any solutions $\varphi(t,\lambda)$ of the equation (1.3),for all$\lambda \in \mathbb{C}$.

Proof: For$0 \leq \sigma < 1$, the proof is the same up to (4.5). In this case (4.5) becomes

$$\left|\varphi^{[r]}(t,\lambda)\right| \leq \sum_{j=1}^{n} (C_j + |\alpha_j(\lambda)|)\left|\varphi_j^{[r]}(t,\lambda_0)\right| + \frac{1}{|1-u|}|\lambda-\lambda_0| \sum_{j,k=1}^{n}\sum_{i=1}^{n-1}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

$$\times \int_0^t \left|\overline{\varphi_k^+(t,\lambda_0)}\right| h_i(s)\left|\varphi^{[i]}(s,\lambda)\right|^\sigma w(s)ds, \quad r = 0,1,\ldots,n-1. \tag{4.8}$$

On applying the Cauchy-Schwartz inequality to the integral in (4.8) we get

$$\int_0^t \left|\overline{\varphi_k^+(t,\lambda_0)}\right| |h_i(s)|\left|\varphi^{[i]}(s,\lambda)\right|^\sigma w(s)ds \leq \left(\int_0^t \left|\overline{\varphi_k^+(s,\lambda_0)}\right|^2 |h_i(s)|^\mu w(s)ds\right)^{\frac{1}{\mu}} \left(\int_0^t \left|\varphi^{[i]}(s,\lambda)\right|^2 w(s)ds\right)^{\frac{\sigma}{2}}, \tag{4.9}$$

where$\mu = 2/(2-\sigma)$.Since$\varphi_k^+(t,\lambda_0) \in L_w^2(0,b)$ for some $\lambda_0 \in \mathbb{C}$, $k = 1,2,\ldots,n$ and $h_i(s)\epsilon L_w^{2/(1-\sigma)}(0,b)$ by hypothesis, then we have$\varphi_k^+(s,\lambda_0)|h_i(s)| \in L_w^\mu(0,b)$, for some $\lambda_0 \in \mathbb{C}$, $k = 1,2,\ldots,n$.Using this fact and (4.9), we obtain

$$\left|\varphi^{[r]}(t,\lambda)\right| \leq \sum_{j=1}^{n} (C_j + |\alpha_j(\lambda)|)\left|\varphi_j^{[r]}(t,\lambda_0)\right| + \frac{1}{|1-u|}K_0|\lambda-\lambda_0| \sum_{j,k=1}^{n}\sum_{i=0}^{n-1}\left|\xi^{jk}\right|\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

$$\times \left(\int_0^t \left|\varphi^{[i]}(s,\lambda)\right|^2 w(s)ds\right)^{\frac{\sigma}{2}}, \quad r = 0,1,\ldots,n-1. \tag{4.10}$$

where$K_0 = \|\varphi_k^+(t,\lambda_0)h(t)\|_\mu$,$\|.\|_\mu$ denotes the norm in $L_w^\mu(0,b)$.By using the inequality

$$(u+v)^2 \leq 2(u^2+v^2), \tag{4.11}$$

implies that

$$\left|\varphi^{[r]}(t,\lambda)\right|^2 \leq 4\sum_{j=1}^{n}\left(C_j^2+\left|\alpha_j(\lambda)\right|^2\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right|^2+\frac{1}{(1-u)^2}4K_0^2|\lambda-\lambda_0|^2\sum_{j,k=1}^{n}\sum_{i=1}^{n-1}\left|\xi^{jk}\right|^2\left|\varphi_j^{[r]}(t,\lambda_0)\right|^2$$

$$\times\left(\int_0^t\left|\varphi^{[i]}(s,\lambda)\right|^2 w(s)ds\right)^{\sigma},\quad r=0,1,\dots,n-1. \tag{4.12}$$

Setting $K_1=\int_0^t\left|\varphi_j^{[r]}(t,\lambda_0)\right|^2 w(s)ds$ for some $\lambda_0\in\mathbb{C}$ and some $r=0,\dots,n-1;$ and integrating (4.12) we obtain

$$\int_0^t\left|\varphi^{[r]}(t,\lambda)\right|^2 w(s)ds \leq K_2+\frac{1}{(1-u)^2}4K_0^2|\lambda-\lambda_0|^2\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}\left|\xi^{jk}\right|^2\int_0^t\left|\varphi_j^{[r]}(s,\lambda_0)\right|^2$$

$$\times\left[\left(\int_0^s\left|\varphi^{[i]}(x,\lambda)\right|^2 w(x)dx\right)^{\sigma}\right]w(s)ds, \tag{4.13}$$

where $K_2=4\sum_{j=1}^{n}\left(C_j^2+\left|\alpha_j(\lambda)\right|^2\right)K_1$.

An application of Lemma 2.4 to (4.12) for $0\leq\sigma<1$ and of Gronwall's inequality to (4.13) for $\sigma=1$ yields the result.

Theorem 4.6: Suppose that F satisfies (4.1) with $0\leq\sigma<1$, $S^r(\tau)\cup S(\tau^+)\subset L_w^2(0,b)\cap L^\infty(0,b)$, for some $\lambda_0\in\mathbb{C}$ and some $r=0,1,\dots,n-1$, and that

$(i)\, k(t)\in L_w^2(0,b)$ for all $t\in[0,b)$,

$(ii)\, h_i(t)\in L_w^p(0,b)$ for some $p,\ 1\leq p\leq 2/(1-\sigma),\ i=0,1,\dots,n-1$. Then $\varphi^{[r]}(t,\lambda)\in L_w^2(0,b)\cap L^\infty(0,b)$, $r=0,1,\dots,n-1$ for any solution $\varphi(t,\lambda)$ of the equation (1.3), for all $\lambda\in\mathbb{C}$.

Proof: Since $S^r(\tau)\cup S(\tau^+)\subset L_w^2(0,b)$ for some $\lambda_0\in\mathbb{C}$ and some $r=0,1,\dots,n-1$, then $\varphi_j^{[r]}(t,\lambda_0),\ \varphi_k^+(s,\lambda_0)\in L_w^q(0,b),\ j,k=1,\dots,n$ for every $q\geq 2$ and for some $\lambda_0\in\mathbb{C}$ and some $r=0,1,\dots,n-1$.

First, suppose that $h_i(t)\in L_w^p(0,b)$ for some $p,\ 1\leq p\leq 2$. Setting

$K_0=\left\|\varphi_j^{[r]}(t,\lambda_0)\right\|_\infty$ and $K_1=\|\varphi_k^+(t,\lambda_0)\|_\infty; j,k=1,\dots,n,$

for some $\lambda_0\in\mathbb{C}$ and some $r=0,1,\dots,n-1$, we have from (4.8) that

$$\left|\varphi^{[r]}(t,\lambda)\right|\leq K_0\sum_{j=1}^{n}\left(C_j+\left|\alpha_j(\lambda)\right|\right)+\frac{1}{|1-u|}K_0K_1|\lambda-\lambda_0|$$

$$\times\left(\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}\left|\xi^{jk}\right|\int_0^t h_i(s)\left|\varphi^{[i]}(s,\lambda)\right|^{\sigma}w(s)ds\right). \tag{4.14}$$

Since $h_i(t)\in L_w^p(0,b)$ for some $p,1\leq p\leq 2$, then Lemma 2.2 together with Gronwall's inequality implies that $\varphi^{[r]}(t,\lambda)\in L^\infty(0,b)$ for all $\lambda\in\mathbb{C}$, i.e., there exists a positive constant K_3 such that

$$\left|\varphi^{[r]}(t,\lambda)\right|\leq K_3 \text{ for all } \lambda\in\mathbb{C},\ t\in[0,b),\ r=0,1,\dots,n-1. \tag{4.15}$$

From (4.8) and (4.15) we obtain

$$\left|\varphi^{[r]}(t,\lambda)\right|\leq\sum_{j=1}^{n}\left(C_j+\left|\alpha_j(\lambda)\right|+K_3\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

for any appropriate constant K_3. Since $\varphi_j^{[r]}(t,\lambda_0) \in L_w^2(0,b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0,1,\dots,n-1$, then$\varphi^{[r]}(t,\lambda) \in L_w^p(0,b)$for all $\lambda \in \mathbb{C}$, $1 \le p \le 2$.Next, suppose that $h_i(t) \in L_w^p(0,b)$ for some p, $2 < p \le 2/(1-\sigma)$, $i = 0,1, \dots, n-1$. Define $q \ge 2$ by

$$\frac{1}{q} = \frac{2-\sigma}{2} - \frac{1}{p}.$$

(which is possible because of the restriction on p). Thus $\varphi_j^{[r]}(s,\lambda_0)$, $\varphi_k^+(t,\lambda_0) \in L_w^q(0,b)$ and $\varphi_k^+(t,\lambda_0)h(t) \in L_w^\mu(0,b)$, $\mu = 2/(2-\sigma)$.

Repeating the same argument in the proof of Theorem 4.5 and from (4.9) to (4.13), we obtain that$\varphi^{[r]}(t,\lambda) \in L_w^2(0,b)$. Returning to (4.9), we find that the integral on the left-hand side is bounded, which implies, by (4.8) that

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^{n} \left(C_j + \left|\alpha_j(\lambda)\right| + K_3\right)\left|\varphi_j^{[r]}(t,\lambda_0)\right|$$

for an appropriate constant K_3. Since $\varphi_j^{[r]}(t,\lambda_0) \in L^\infty(0,b)$, this completes the proof. We refer to [1 - 6] and [15] for more details.

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